1.- Sean $\bar{a}$, $\bar{b}$ y $\bar{c}$ vectores tales que: $\bar{a}+\bar{b}+\bar{c}=0$, $|\bar{a}|=3$, $|\bar{b}|=1$
$|\bar{c}|=4$.
Determinar $\bar{a}.\bar{b}+\bar{a}.\bar{c}+\bar{b}.\bar{c}$.

$$\rightarrow \quad (\bar{a}+\bar{b}+\bar{c}=0)^2$$

$$|\bar{a}|^2+|\bar{b}|^2+|\bar{c}|^2+2(\bar{a}.\bar{b}+\bar{b}.\bar{c}+\bar{a}.\bar{c})=0$$
$$3^2+1^2+4^2+2(\bar{a}.\bar{b}+\bar{b}.\bar{c}+\bar{a}.\bar{c})=0$$
$$26+2(\bar{a}.\bar{b}+\bar{b}.\bar{c}+\bar{a}.\bar{c})=0$$

$$\rightarrow \quad \bar{a}.\bar{b}+\bar{b}.\bar{c}+\bar{a}.\bar{c}=-13.$$

2.- Sean $\bar{a}$, $\bar{b}$ y $\bar{c}$ vectores, demostrar que $(\bar{a}.\bar{c})b-(\bar{a}.\bar{b})\bar{c}$ es perpendicular al vector $\bar{a}$.

$$\rightarrow \quad ((\bar{a}.\bar{c})\bar{b}-(\bar{a}.\bar{b})\bar{c}).a=0$$

$$(\bar{a}.\bar{c})(\bar{b}.\bar{a})-(\bar{a}.\bar{b})(\bar{c}.\bar{a})=0$$

Como $\bar{a}.\bar{c}=c.\bar{a}$ y $b.\bar{a}=\bar{a}.b$

entonces $(\bar{a}.\bar{c})(\bar{b}.\bar{a})-(\bar{b}.\bar{a})(\bar{a}.\bar{c})=0$

$$0=0 \checkmark$$

$\therefore$ $(\bar{a}.\bar{c})\bar{b}-(\bar{a}.\bar{b})\bar{c}$ es perpendicular al vector $\bar{a}$.

3.- Demostrar que: $\bar{b} - \frac{(\bar{a}.\bar{b})}{|\bar{a}|^2}\bar{a}$ es perpendicular al vector $\bar{a}$.

$$\rightarrow \quad \left(\bar{b} - \frac{(\bar{a}.\bar{b})}{|\bar{a}|^2}\bar{a}\right) . \bar{a} = 0$$

$$\bar{b}.\bar{a} - \frac{(\bar{a}.\bar{b})(\bar{a}.\bar{a})}{|\bar{a}|^2} = 0$$

$$\bar{b}.\bar{a} - \frac{(\bar{a}.\bar{b})}{|\bar{a}|^2}|\bar{a}|^2 = 0$$

$$\bar{b}.\bar{a} - \bar{a}.\bar{b} = 0$$

$$\bar{a}.\bar{b} = \bar{b}.\bar{a} \quad \checkmark$$

$\therefore \quad \bar{b} - \frac{(\bar{a}.\bar{b})}{|\bar{a}|^2}\bar{a}$ es perpendicular al vector $\bar{a}$.

4.- Determinar: $2\bar{c}.\bar{d}$, si $\bar{a}+\bar{b}+\bar{c}+\bar{d} = 0$; $|\bar{a}+\bar{b}| = 6$; $|\bar{c}| = 3$; $|\bar{d}| = 4$.

DATO: $\bar{a}+\bar{b}+\bar{c}+\bar{d} = 0$

$$\bar{a}+\bar{b} = -\bar{c}-\bar{d}$$

$$(|\bar{a}+\bar{b}| = |-\bar{c}-\bar{d}|)^2$$

$$6^2 = |\bar{c}|^2 + |\bar{d}|^2 + 2\bar{c}.\bar{d}$$

$$36 = 3^2 + 4^2 + 2\bar{c}.\bar{d}$$

$$36 = 9 + 16 + 2\bar{c}.\bar{d}$$

$$36 = 25 + 2\bar{c}.\bar{d}$$

$$2\bar{c}.\bar{d} = 11$$

5. Sean $\bar{a},\bar{b}$ vectores de $\mathbb{R}^3$ demostrar que:

$\|\bar{a}\times\bar{b}\| \leq \|\bar{a}\|\|\bar{b}\|$

$|\bar{a}\times\bar{b}| = |\bar{a}||\bar{b}| \operatorname{sen}\alpha \quad \ldots (I)$

Sabemos que: $-1 \leq \operatorname{sen}\alpha \leq 1$

$\rightarrow -|\bar{a}||\bar{b}| \leq |\bar{a}||\bar{b}| \operatorname{sen}\alpha \leq |\bar{a}||\bar{b}| \quad \ldots (II)$

Remplazamos (I) en (II):

$|\bar{a}\times\bar{b}| \leq |\bar{a}||\bar{b}|$ ✓

6. Determinar $|(3\bar{a}-\bar{b})\times(\bar{a}-2\bar{b})|$, si $\bar{a}$ es ortogonal a $\bar{b}$ y $|\bar{a}|=3$, $|\bar{b}|=4$

$|(3\bar{a}-\bar{b})\times(\bar{a}-2\bar{b})|$

$= |(3\bar{a}-\bar{b})\times\bar{a} - (3\bar{a}-\bar{b})\times 2\bar{b}|$

$= |(3\bar{a})\times\bar{a} - \bar{b}\times\bar{a} + 2\bar{b}\times(3\bar{a}-\bar{b})|$

$= |3\underbrace{(\bar{a}\times\bar{a})}_{0} + \bar{a}\times\bar{b} + 6(\bar{b}\times\bar{a}) - 2\underbrace{(\bar{b}\times\bar{b})}_{0}|$

$= |\bar{a}\times\bar{b} + 6(\bar{b}\times\bar{a})|$

$= |-(\bar{b}\times\bar{a}) + 6(\bar{b}\times\bar{a})|$

$= |5(\bar{b}\times\bar{a})|$

$= 5|\bar{b}\times\bar{a}|$

$= 5.|b||a| \operatorname{sen}\alpha$

$= 5.4.3.\underbrace{\operatorname{sen}\frac{\pi}{2}}_{1}$

$= 60$

$\bar{a}.\bar{b} = 0$, $|a|=3$, $|\bar{b}|=4$

$\bar{a}.\bar{b} = |a||b|\cos\alpha$

$0 = 3.4\cos\alpha$

$0 = \cos\alpha$

Como $\alpha \in [0,\pi]$

$\rightarrow \boxed{\alpha = \frac{\pi}{2}}$

7. Sean $\bar{a},\bar{b}$ vectores de $\mathbb{R}^3$, determinar:

$$|\bar{a}\times\bar{b}| = |\bar{a}||\bar{b}| \leftrightarrow \bar{a}\cdot\bar{b} = 0$$

Sea: $\bar{a} = (a_1, a_2, a_3) \wedge \bar{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$

Tenemos que: $\bar{a}\cdot\bar{b} = 0$

$$\rightarrow (a_1, a_2, a_3)\cdot(b_1, b_2, b_3) = 0$$

$$\rightarrow a_1b_1 + a_2b_2 + a_3b_3 = 0 \quad \ldots (*)$$

Desarrollando vemos que:

$$\rightarrow |\bar{a}||\bar{b}| = \sqrt{a_1^2+a_2^2+a_3^2}\cdot\sqrt{b_1^2+b_2^2+b_3^2}$$
$$= \sqrt{(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)}$$

$$\rightarrow \bar{a}\times\bar{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1)$$

$$|\bar{a}\times\bar{b}| = \left((a_2b_3-a_3b_2)^2 + (a_3b_1-a_1b_3)^2 + (a_1b_2-a_2b_1)^2\right)^{1/2}$$

$$|\bar{a}\times\bar{b}| = \left(a_2^2b_3^2 + a_3^2b_2^2 - 2a_2b_3a_3b_2 + a_3^2b_1^2 + a_1^2b_3^2 - 2a_3b_1a_1b_3 + a_1^2b_2^2 + a_2^2b_1^2 - 2a_1b_2a_2b_1\right)^{1/2}$$

$$|\bar{a}\times\bar{b}| = \left(a_2^2b_3^2 + a_3^2b_2^2 + a_3^2b_1^2 + a_1^2b_3^2 + a_1^2b_2^2 + a_2^2b_1^2 - (a_2b_2a_3b_3 + a_1b_1a_3b_3 + a_1b_1a_2b_2 + a_2b_2a_3b_3 + a_1b_1a_3b_3 + a_1b_1a_2b_2)\right)^{1/2}$$

$$|\bar{a}\times\bar{b}| = \left(a_2^2b_3^2 + a_3^2b_2^2 + a_3^2b_1^2 + a_1^2b_3^2 + a_1^2b_2^2 + a_2^2b_1^2 - a_3b_3(a_2b_2+a_1b_1) - a_2b_2(a_1b_1+a_3b_3) - a_1b_1(a_3b_3+a_2b_2)\right)^{1/2}$$

Con $(*)$ tendremos:

$$|a\times b| = \sqrt{a_1^2b_1^2 + a_1^2b_2^2 + a_1^2b_3^2 + a_2^2b_1^2 + a_2^2b_2^2 + a_2^2b_3^2 + a_3^2b_1^2 + a_3^2b_2^2 + a_3^2b_3^2}$$

$$|a\times b| = \sqrt{(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)}$$

$$\therefore |\bar{a}\times\bar{b}| = |\bar{a}||\bar{b}|$$

8) Sean $\bar{A}, \bar{B}$ y $\bar{C}$ vectores en $\mathbb{R}^3$ determinar

$$(\bar{A}\times\bar{B})\times\bar{C} + (\bar{B}\times\bar{C})\times\bar{A} + (\bar{C}\times\bar{A})\times\bar{B} + \underbrace{(\bar{A}\times\bar{A})}_{0}\cdot(\bar{A}\times\bar{B})$$

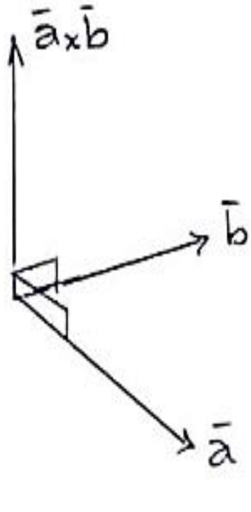
$$= (A\cdot C)B - (B\cdot C)A + (B\cdot A)C - (C\cdot A)B + (C\cdot B)A - (A\cdot B)C + 0$$

$$= (A\cdot C)B - (B\cdot C)A + (B\cdot A)C - (A\cdot C)B + (B\cdot C)A - (B\cdot A)C$$

$$= \underbrace{(A\cdot C)B - (A\cdot C)B}_{0} + \underbrace{(B\cdot C)A - (B\cdot C)A}_{0} + \underbrace{(B\cdot A)C - (B\cdot A)C}_{0}$$

$$= 0$$

9.- Sean $\vec{a}, \vec{b}$ y $\vec{c}$ vectores, determinar si el conjunto $\{\bar{a}\times\bar{b}, \bar{a}, \bar{b}\}$ es L.D o es L.I.



$\bar{a}\times\bar{b} \perp \bar{a} \rightarrow (a\times\bar{b})\cdot\bar{a} = 0$

$\bar{a}\times\bar{b} \perp \bar{b} \rightarrow (\bar{a}\times\bar{b})\cdot\bar{b} = 0$

$\therefore \{\bar{a}\times\bar{b}, \bar{a}, \bar{b}\}$ son linealmente independientes.

10.- Sean $\bar{a}, \bar{b}$ y $\bar{c}$ vectores no paralelos entre sí, determinar si el conjunto $\{ proy_{\bar{b}} \bar{a} \, ; \, proy_{\bar{a}} \bar{b} \, ; \, proy_{\bar{a}} \bar{c} \}$ es L.D o L.I.

Si $[m \; n \; p] = 0 \rightarrow \{m, n, p\}$ son L.D

$$proy_{\bar{b}} \bar{a} = \frac{(\bar{a} \cdot \bar{b}) \bar{b}}{|\bar{b}|^2} \; ; \; proy_{\bar{a}} \bar{b} = \frac{(\bar{a} \cdot \bar{b}) \bar{a}}{|\bar{a}|^2} \; ; \; proy_{\bar{a}} \bar{c} = \frac{(\bar{a} \cdot \bar{c}) \bar{a}}{|\bar{a}|^2}$$

$$[proy_{\bar{b}} \bar{a} \;\; proy_{\bar{a}} \bar{b} \;\; proy_{\bar{a}} \bar{c}] = proy_{\bar{b}} \bar{a} \cdot (proy_{\bar{a}} \bar{b} \times proy_{\bar{a}} \bar{c})$$

$$= \frac{(\bar{a} \cdot \bar{b}) \bar{b}}{|\bar{b}|^2} \cdot \left(\frac{(\bar{a} \cdot \bar{b}) \bar{a}}{|\bar{a}|^2} \times \frac{(\bar{a} \cdot \bar{c}) \bar{a}}{|\bar{a}|^2} \right)$$

$$\left. = \frac{(\bar{a} \cdot \bar{b}) \bar{b}}{|\bar{b}|^2} \cdot \left(\frac{(\bar{a} \cdot \bar{b})(\bar{a} \cdot \bar{c})}{|\bar{a}|^4} (\bar{a} \times \bar{a}) \right) \right\} \quad \bar{a} \times \bar{a} = 0$$

$$[proy_{\bar{b}} \bar{a} \;\; proy_{\bar{a}} \bar{b} \;\; proy_{\bar{a}} \bar{c}] = 0$$

$\therefore \{ proy_{\bar{b}} \bar{a} \, ; \, proy_{\bar{a}} \bar{b} \, ; \, proy_{\bar{a}} \bar{c} \}$ son L.D

11.- Si $proy_{\bar{b}} \bar{a} = (7,3,5)$ y $proy_{\bar{a}} \bar{b} = (-8,4,2)$, hallar los vectores $\bar{a}$ y $\bar{b}$.

$$proy_{\bar{b}} \bar{a} = (7,3,5) \longrightarrow \bar{b} = m(7,3,5)$$

$$proy_{\bar{a}} \bar{b} = (-8,4,2) \longrightarrow \bar{a} = t(-8,4,2)$$

$$\bar{a} \cdot \bar{b} = (7m)(-8t) + (3m)(4t) + (5m)(2t) = -34mt$$

$$|\bar{a}|^2 = (-8t)^2 + (4t)^2 + (2t)^2 = 84t^2 \; ; \; |\bar{b}|^2 = (7m)^2 + (3m)^2 + (5m)^2 = 83m^2$$

$$\text{proy}_{\bar b}\,\bar a = \frac{(\bar a\cdot\bar b)\bar b}{|\bar b|^2} = (7,3,5)$$

$$\rightarrow \frac{(\bar a\cdot\bar b)b_1}{|\bar b|^2} = 7$$

$$\frac{(-34mt)(7m)}{83m^2} = 7$$

$$\boxed{t = \frac{-83}{34}}$$

$$\text{proy}_{\bar a}\,\bar b = \frac{(\bar b\cdot\bar a)\bar a}{|\bar a|^2} = (-8,4,2)$$

$$\rightarrow \frac{(\bar b\cdot\bar a)a_1}{|\bar a|^2} = -8$$

$$\frac{(-34mt)(-8t)}{84t^2} = -8$$

$$\boxed{m = \frac{-84}{34}}$$

$$a = -\frac{83}{84}(-8,4,2) = \left(\frac{166}{21};\ -\frac{83}{21};\ -\frac{83}{42}\right)$$

$$b = -\frac{84}{34}(7,3,5) = \left(\frac{-588}{34};\ \frac{-252}{34};\ \frac{-420}{34}\right)$$

12.- Sean $\vec A$, $\vec B$ y $\vec C$ vectores R^3, demostrar:

$$(\bar A\times\bar B)\cdot(\bar C\times\bar D) = (\bar A\cdot\bar C)(\bar B\cdot\bar D) - (\bar A\cdot\bar D)(\bar B\cdot\bar C)$$

$$(\bar A\times\bar B)\cdot(\bar C\times\bar D) = ((\bar C\times\bar D)\times\bar A)\cdot\bar B$$

$$= [(\bar A\cdot\bar C)\bar D - (\bar A\cdot\bar D)\bar C]\cdot\bar B$$

$$= (\bar A\cdot\bar C)(\bar D\cdot\bar B) - (\bar A\cdot\bar D)(\bar C\cdot\bar B)$$

$$(\bar A\times\bar B)\cdot(\bar C\times\bar D) = (\bar A\cdot\bar C)(\bar B\cdot\bar D) - (\bar A\cdot\bar D)(\bar B\cdot\bar C)$$ l.q.q.d

(13)

a) Si $\vec{a} = \vec{p} \times \vec{n}$

$b = \vec{q} \times \vec{n}$ ES COPLANARES

$\vec{c} = \vec{r} \times \vec{n}$ si $[a\ b\ c] = 0$

(VERDADERO)

$\vec{a} \cdot (\vec{b} \times \vec{c})$

$\vec{b} \perp \vec{n} \wedge \vec{c} \perp \vec{n}$

$\rightarrow \vec{b} \parallel \vec{c} \perp \vec{n}$

$\vec{b} \times \vec{c} \parallel \vec{n} \wedge a \perp \vec{n}$

$\rightarrow a \perp (b \times c)$

$\therefore\ a \cdot (b \times c) = 0$

$[a\ b\ c] = 0$

b) Si $\vec{a} + \vec{b} + \vec{c} = \vec{0} \Rightarrow \vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$

(VERDADERO)

$\vec{a} \times (\vec{a} + \vec{b} + \vec{c}) = \vec{a} \times \vec{0}$

$\vec{a} \times \vec{a} + \vec{a} \times \vec{b} + \vec{a} \times \vec{c} = \vec{0}$

$\vec{a} \times \vec{b} = -\vec{a} \times \vec{c}$

$\vec{a} \times \vec{b} = \vec{c} \times \vec{a}$

$\vec{b} \times (\vec{a} + \vec{b} + \vec{c}) = \vec{b} \times \vec{0}$

$\vec{b} \times \vec{a} + \vec{b} \times \vec{b} + \vec{b} \times \vec{c} = \vec{0}$

$\vec{b} \times \vec{a} = -\vec{b} \times \vec{c}$

$\vec{b} \times \vec{c} = -\vec{b} \times \vec{a}$

$\boxed{\vec{b} \times \vec{c} = \vec{a} \times \vec{b}}$

$\therefore\ \vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$

l.q.q.

c) Si $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$ y $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$

$\Rightarrow$ $\vec{a} - \vec{d}$ y $\vec{b} - \vec{c}$ son colineales

(VERDADERO)

$$\vec{a} \times \vec{b} + b \times d = c \times d + \vec{a} \times \vec{c}$$

$$\vec{a} \times \vec{b} - \vec{d} \times \vec{b} = -\vec{d} \times \vec{c} + \vec{a} \times \vec{c}$$

$$(\vec{a} - \vec{d}) \times \vec{b} = (\vec{a} - \vec{d}) \times \vec{c}$$

$$(\vec{a} - \vec{d}) \times \vec{b} - (\vec{a} - \vec{d}) \times \vec{c} = \vec{0}$$

$$(\vec{a} - \vec{d}) \times (\vec{b} - \vec{c}) = 0$$

angulo entre $(a - \vec{d})$ y

$(\vec{b} - \vec{c})$ es cero

o $180°$

entonces son colineales.

d) Si $\vec{a} \perp \vec{b} \Rightarrow \vec{a} \times (\vec{a} \times \underbrace{(\vec{a} \times (\vec{a} \times \vec{b}))}) \neq |a|^4 \vec{b}$

(VERDADERO)

$$\vec{a}(\underbrace{\vec{a} \cdot \vec{b}}) - \vec{b}(a \cdot a)$$

$$\vec{a} \times (\vec{a} \times (\vec{a} \cdot 0 - \vec{b}))$$

$$\vec{a} \times (\vec{a} \times (-\vec{b}))$$

$$\vec{a}(\underbrace{\vec{a} \cdot (-\vec{b})}) - (-\vec{b})(\vec{a} \cdot \vec{a}$$

$$\underbrace{\vec{a} \cdot 0 - (-\vec{b}) \cdot 1}$$

$$\vec{b} \neq |a|^4 \vec{b}$$

(14)

$A(4,0,-1)$. $B(a,b,0)$ y C

$\vec{PB} = \text{proy}_{\vec{AB}} \vec{CB} = (3;-6;3)$

$\vec{PB} \parallel \vec{AB} \rightarrow t(3,-6;3) = (a-4,b,1)$

$\vec{PC} = C-P = (1;-3,-7)$

$(3t,-6t;3t) = (a-4,b,1)$

$B(5,-2;0)$ $\quad 3t=1 \quad 3t=a-4 \quad -6t=b$

$\quad t=1/3 \quad a=5 \quad b=-2$

$\vec{AB} = -\vec{BA}$

$\vec{BA} = A-B = \overset{(4,0;-1)}{}-(5,-2;0) = (-1;2;-1)$

$\vec{BC} = C-B = C-(a,b,0)$

$\vec{PB} = B-P = (3,-6;3)$

$(5,-2;0) - P = (3;-6;3)$

$P = (2;4;-3)$

$\vec{PC} = C-P = (1,-3;-7)$

$C = (2,4;-3)+(1;-3;-7) = (3;1;-10)$

$\vec{BC} = C-B = (3;1;-10)-(5;-2;0)$

$\vec{BC} = (-2;3;-10)$

$$\vec{BC} = \begin{vmatrix} i & j & k \\ -1 & 2 & -1 \\ -2 & 3 & -10 \end{vmatrix} = -17i-8j+k$$

$-4k \quad -20i$

$-3i \quad i \quad j \quad k \quad -3k$

$10j \quad -1 \quad 2 \quad -1 \quad 2j$

$\vec{BQ} = Q-B = (-17;-8;1)$

$Q = (5;-2;0)+(-17;-8;1)$

$Q = (-12;-10;1)$

$\overrightarrow{AC} = C - A = (3; 1; -10) - (4; 0; -1) = (-1; 1; -9)$

$\overline{PQ} = Q - P = (-12, -10; 1) - (2, 4; -3) = (-14, -14; 4)$

$$\cos\alpha = \frac{(-1, 1, -9) \cdot (-14, -14, 4)}{|(-1, 1, 9)|\,|(-14, -14, 4)|} = 14 - 14 - 36$$

$$\cos\alpha = \frac{-36}{\sqrt{83} \cdot \sqrt{408}}$$

$$\alpha = \arccos\left(\frac{-18}{\sqrt{83} \cdot \sqrt{102}}\right)$$